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Further comments on algebras for external electromagnetic interaction with relativistic fields

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Abstract. An algebra providing causal minimal interaction with an external electromagnetic field is considered. It was incorrectly claimed in an earlier paper that this algebra was infinite and here it is shown to be finite, but not large enough to support a theory with spin greater than one. Also, a method is given for finding the size of some Young symmetrizer algebras, which may provide further examples of causal theories.

1. Introduction

In a previous paper (Cox 1976) we have proposed a method for finding possible causal interacting high-spin theories. The theories are based on the usual linear field equation

$$(\Gamma_\mu p^\mu + \mathrm{i}\chi)\psi = 0 \qquad (1.1)$$

with minimal coupling to an external electromagnetic field by the replacement

$$p_\mu \to \pi_\mu = p_\mu + eA_\mu .$$

As a sufficient condition for causality it is required that the true equation of motion, in the interaction case, should have a principal part which is Klein–Gordon. The resulting conditions on the Γ_μ are given in Young symmetrizer form:

$$(Y^{(i)}_{F_j} M)_{\mu\nu\rho\ldots\epsilon} = 0 \qquad \text{for some } i, j \qquad (1.2)$$

where $Y^{(i)}_{F_j}$ is the Young symmetrizer corresponding to the ith standard tableau of the Young frame F_j, and

$$M_{\mu\nu\rho\ldots\epsilon} = (\Gamma_\mu\Gamma_\nu - \delta_{\mu\nu})\Gamma_\rho \ldots \Gamma_\epsilon . \qquad (1.3)$$

In particular, it was shown that the fourth rank algebra

$$\begin{aligned} &(Y_{\square\square\square\square}\, M)_{\mu\nu\rho\alpha} = 0 \\ &(Y^{(i)}_{\substack{\square\square\square\\ \square}}\, M)_{\mu\nu\rho\alpha} = 0 \qquad i = 1, 2, 3 \\ &(Y^{(i)}_{\substack{\square\square\\ \square\\ \square}}\, M)_{\mu\nu\rho\alpha} = 0 \qquad i = 1, 2, 3 \end{aligned} \qquad (1.4)$$

yields a causal theory, and that this algebra was infinite—that is, the requirement of causality was not sufficient to generate a finite algebra. Unfortunately, the proof that the algebra was infinite is incorrect, and in fact, as we prove later in this paper the algebra

(1.4) is *finite*. However, as we also show, the algebra (1.4) is not big enough to support a high-spin theory—that is, a theory with spin greater than one. The same of course applies to any sub-algebra, such as that suggested in the previous paper, obtained by imposing further relations of the type

$$(Y^{(i)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} M)_{\mu\nu\rho\alpha} = 0$$

or

$$(Y_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}} M)_{\mu\nu\rho\alpha} = 0.$$

An example of the latter type of sub-algebra is treated separately below, in equation (2.14), in order to illustrate an approach to Young symmetrizer algebras such as (1.2), which seems to be reasonably fruitful for a particular class of algebras. The algebra (1.4) does not belong to this class, and it was a failure to notice this which led to the erroneous conclusion that it was an infinite algebra. We analyse the more general algebra (1.4) separately in § 3.

The type of interaction theory we are considering here—one in which the principal part of the true equation of motion is Klein–Gordon—is a particular case of the 'nice interactions' of Bellisard (1976), who has rigorously proved their causal nature. Bellisard has posed the question of whether field equations exist describing spin greater than or equal to $\frac{3}{2}$ admitting nice interactions. Our results seem to show that the nice minimal electromagnetic interaction with a field theory based on (1.1), governed by a fourth rank tensor algebra for the Γ_μ, cannot be high spin. We hope to generalize these results to higher rank algebra in the future, to at least partially answer Bellisard's question.

The general approach in this paper is to find the size of the algebra, i.e. the maximum possible number of independent elements, and to show that this is insufficient to support a good causal high-spin theory. We draw attention to the fact that high-spin algebras have to be very large, even for the spin-$\frac{3}{2}$ case, which requires at least 400 elements (the Rarita–Schwinger spin-$\frac{3}{2}$ theory requires only 256 elements, but is acausal).

2. The analysis of some simple Young symmetrizer algebras

The analysis of algebras described by relations of type (1.4) seems to be very difficult, even in the initial stages of finding the number of independent elements in the algebra. The only such algebras known in detail are the Dirac and Duffin–Kemmer (DK) algebras, which satisfy

$$(Y_{\square\square} M)_{\mu\nu} = 0 \qquad \text{(Dirac)} \tag{2.1}$$

$$\left.\begin{aligned}(Y_{\square\square\square} M)_{\mu\nu\rho} &= 0\\ (Y_{\begin{smallmatrix}\boxed{1}&\boxed{3}\\\boxed{2}&\end{smallmatrix}} M)_{\mu\nu\rho} &= 0\end{aligned}\right\} \quad \text{(DK)} \tag{2.2}$$

respectively (also see (2.6), below). The numbers 1, 2, 3 in the frame correspond to the indices μ, ν, ρ respectively. These algebras have 16 and 125 independent elements respectively. The methods used to analyse these algebras are rather *ad hoc* and do not conveniently generalize to the algebra (1.4) for example.

As a preliminary to studying the field theories based on such algebras as (1.2) it is necessary to know whether the algebra is finite. If so, then the size of the algebra will give information on the maximum spin which the corresponding theory can accommodate. Unfortunately it seems difficult to calculate the independent products in a general Young symmetrizer algebra. However, when the algebra is equivalent to a group of simple symmetry conditions on $M_{\mu\nu\rho\ldots\epsilon}$, then we can calculate the number of independent products for each rank, using a method given in the previous paper (Cox 1976). While the DK algebra (2.2) is easily dealt with more directly, we will use it here as a simple illustration of the procedure.

For the DK algebra we can write

$$M_{\mu\nu\rho}=\frac{1}{3!}\left(Y_{\square\square\square}+2Y_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}+2Y_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}+Y_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\right)M_{\mu\nu\rho}$$

using the resolution of the identity in terms of Young symmetrizers,

$$M_{\mu\nu\rho}=\frac{1}{3!}\left(2Y_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}+Y_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\right)M_{\mu\nu\rho} \tag{2.3}$$

using (2.2). The group of all permutations under which the last expression is either invariant or changes sign only is given by

$$G=\{e,(13)^{(-)}\} \tag{2.4}$$

where e is the identity permutation and the superscript $(-)$ means that the expression changes sign under the permutation (13) of the indices. The algebra (2.2) therefore implies that the third rank tensor $M_{\mu\nu\rho}$ has symmetry $\{e,(13)^{(-)}\}$ or $\{e,(\mu\rho)^{(-)}\}$. Conversely, it is easy to verify directly in this case that if $M_{\mu\nu\rho}$ has symmetry group $\{e,(13)^{(-)}\}$, then it satisfies the algebra (2.2). Thus, the requirement that $M_{\mu\nu\rho}$ has symmetry group (2.4) is completely equivalent to the algebra (2.2). This rewriting of the algebraic relations (2.2) leads to a simplification in the treatment of the algebra.

Consider the third rank product $\Gamma_\mu\Gamma_\nu\Gamma_\rho$ in the DK algebra. Since $M_{\mu\nu\rho}=(\Gamma_\mu\Gamma_\nu-\delta_{\mu\nu})\Gamma_\rho$ is subject to the symmetry (2.4), $\Gamma_\mu\Gamma_\nu\Gamma_\rho$ will also be subject to the same symmetry, except that terms like $\delta_{\mu\nu}\Gamma_\rho$ will arise. For example

$$M_{\mu\nu\rho}=-M_{\rho\nu\mu}$$

is equivalent to

$$\Gamma_\mu\Gamma_\nu\Gamma_\rho=-\Gamma_\rho\Gamma_\nu\Gamma_\mu+\delta_{\mu\nu}\Gamma_\rho+\delta_{\rho\nu}\Gamma_\mu.$$

However, in finding the independent elements of the algebra, the object is to express higher rank products in terms of lower rank products and so, when dealing with third rank products, the terms linear in Γ_μ can be ignored. Thus, we can say that the symmetry (2.4) on $M_{\mu\nu\rho}$ implies the same symmetry ('modulo lower rank products') on the third rank products $\Gamma_\mu\Gamma_\nu\Gamma_\rho$.

Now consider the higher rank products in the DK algebra. Let $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ denote the rth rank product. By virtue of the DK algebra every set of three adjacent indices of $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ possesses the symmetry (2.4)—this can be expressed simply as 'the rth rank product is antisymmetric in alternate indices'. We can therefore write down a set P_r, of $r-1$ permutations: e, under which $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ is invariant, and $(\mu_1\mu_3)^{(-)}$, $(\mu_2\mu_4)^{(-)}$, ... $(\mu_{r-2}\mu_r)^{(-)}$, under which $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ changes sign. The elements of P_r generate a subgroup G_r of S_r, the symmetric group on r objects, and when the signs are taken into account G_r gives the symmetry of the rth rank product implied by the third rank symmetry (2.4), which in turn is equivalent to the DK algebra. No other symmetries are possible for $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ in the DK algebra, because these would have to be consequences of relations other than (2.2) and therefore outside of the DK algebra. Thus, G_r gives the complete symmetry of the rth rank product in the DK algebra. Given the complete symmetry of a tensor, we can calculate the number of independent components, which in this case would be the number of independent rth rank products in the DK algebra. Thus, if $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ has symmetry $G_r=\{p_i\}$ then the number of independent components of $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ in four dimensions is given by (Cox 1976)

$$n_r=\frac{1}{|G|}\sum_{p_i}\delta_{p_i}4^{c_i} \tag{2.5}$$

where $|G|=$ order of G, $\delta_{p_i}=+1\ (-1)$ if $\Gamma_{\mu_1\mu_2\ldots\mu_r}$ is invariant (changes sign) under p_i, and c_i is the number of disjoint cycles in p_i—for example $e=(1)(2)\ldots(r)$, so $c_e=r$. If for some r we obtain $n_r=0$, then there are no independent rth rank products in the algebra—they can all be expressed in terms of lower rank products—and the algebra is finite. In the case of the DK algebra this happens for $r=9$. All products for rank greater than 8 can be expressed in terms of lower rank products. This is a simple consequence of the antisymmetry of DK products in alternate indices, and in fact the method given above would be unnecessarily complicated for the DK algebra. However, it seems to be the only simple approach for some algebras.

For a concrete example, consider the fourth rank products in the DK algebra, $\Gamma_{\mu_1\mu_2\mu_3\mu_4}$. We have

$$P_4=\{e,(13)^{(-)},(24)^{(-)}\}$$

which generates the symmetry group

$$G_4=\{e,(13)^{(-)},(24)^{(-)},(13)(24)\}.$$

Then (2.5) gives

$$n_4=\tfrac{1}{4}(4^4-2\times4^3+4^2)=36.$$

So there are 36 independent fourth rank products in the DK algebra.

For the fifth rank products we have

$$P_5=\{e,(13)^{(-)},(24)^{(-)},(35)^{(-)}\}.$$

The subgroup of S_5 generated is conveniently obtained by noting that P_5 generates the direct product of the subgroup generated by $\{e,(24)^{(-)}\}$ and $\{e,(13)^{(-)},(35)^{(-)}\}$. Thus

$$G_5=\{e,(24)^{(-)}\}\otimes\{e,(13)^{(-)},(35)^{(-)},(15)^{(-)},(135),(153)\}$$

and (2.5) gives

$$n_5=\tfrac{1}{12}(4^5-4\times4^4+5\times4^3-2\times4^2)=24.$$

So there are 24 independent fifth rank products in the DK algebra.

The full list of independent elements in the DK algebra has been given by Kemmer (1939), from which the above results may be verified. The number of higher rank products can be determined similarly, but the working becomes very complicated by the above method.

When the DK algebra is expressed in the Young symmetrizer form (2.2) an obvious and interesting question arises as to the significance of other possibilities such as

$$(Y_{\Box\Box\Box}\ M)_{\mu\nu\rho}=0 \qquad (Y_{\substack{1\,2\\3}}\ M)_{\mu\nu\rho}=0 \tag{2.6}$$

which also are causal and unique mass (Cox 1976). The equivalent symmetry group in this case is

$$G_3=\{e,(12)^{(-)}\}.$$

There are 4 independent first rank products, 16 second rank and 24 third rank products (20 from $Y_{\substack{1\,3\\2}} M_{\mu\nu\rho}$ and 4 from $Y_{\substack{\Box\\\Box\\\Box}} M_{\mu\nu\rho}$ (Hammermesh 1962, p 388)) in this algebra.

For the fourth rank products G_3 generates the symmetry group $G_4=\{e,(12)^{(-)},(23)^{(-)},(13)^{(-)},(123),(132)\}$, giving $n_4=16$. There are 4 fifth rank products, since G_3 implies that the general fifth rank product $\Gamma_{\mu\nu\rho\alpha\epsilon}$ is antisymmetric in any four adjacent indices and so must be reducible to the form $\pm\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_\epsilon$ where ϵ is arbitrary. Similarly, the sixth rank product must be antisymmetric in the first five indices and must therefore be fully reducible to lower rank products. So, in all, the algebra (2.6) has 65 independent elements, if we include the unit element. This is not large enough to yield a spin-1 manifestly covariant Lagrangian field theory based on the first order equation (1.1). The usual DK theory (2.2) is in fact the simplest such theory and this already contains 100 independent elements in its algebra—the Γ_μ are 10×10 matrices. In fact, the algebra (2.6) is the Hermitian conjugate algebra to that obeyed by the matrices β_μ used by Capri (1969) for a new class of spin-$\frac{1}{2}$ equations, and later studied in detail by Santhanam and Tekumalla (1974). For, explicitly, the algebra (2.6) is

$$M_{\mu\nu\rho}+M_{\mu\rho\nu}+M_{\nu\mu\rho}+M_{\nu\rho\mu}+M_{\rho\mu\nu}+M_{\rho\nu\mu}=0 \tag{2.7}$$

$$M_{\mu\nu\rho}+M_{\nu\mu\rho}-M_{\rho\nu\mu}-M_{\nu\rho\mu}=0 \tag{2.8}$$

where $M_{\mu\nu\rho}=(\beta_\mu\beta_\nu-\delta_{\mu\nu})\beta_\rho$. Putting $S_{\mu\nu\rho}=M_{\mu\nu\rho}+M_{\nu\mu\rho}=S_{\nu\mu\rho}$, (2.8) implies $S_{\mu\nu\rho}=S_{\rho\nu\mu}$, while (2.7) can be rewritten $S_{\mu\nu\rho}+S_{\mu\rho\nu}+S_{\nu\rho\mu}=0$. These equations yield $S_{\mu\nu\rho}=0$. Conversely $S_{\mu\nu\rho}=0$ implies (2.6). Thus (2.6) is equivalent to the algebra

$$(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu)\beta_\rho=2\delta_{\mu\nu}\beta_\rho \tag{2.9}$$

which is precisely that studied by Santhanam and Tekumalla.

The method given above can be applied to any sth rank Young symmetrizer algebra which is equivalent to a group of simple symmetry conditions. First, the arbitrary sth rank product is resolved into irreducible symmetry classes using the Young symmetrizers. The general form of the sth rank product in the algebra is then obtained by omitting those symmetrizers appearing in the algebra (1.2). The symmetry of the product can then be found, and must be shown to be completely equivalent to the algebra (1.2). When this is so, the symmetries of the higher rank products in the algebra can be obtained and (2.5) can be used to calculate the corresponding number of independent elements. When the algebra (1.2) is not equivalent to a symmetry group, then the analysis is more complicated, and a general approach seems more difficult to find. The algebra (1.4) is not equivalent to a symmetry group and is treated separately in the next section.

Historically, the Young operators were first introduced by Young precisely for the analysis of tensor equations via the study of *substitutional equations* of the type

$$L_i X = 0 \tag{2.10}$$

where L_i are known permutation operators (i.e. elements of the group algebra of a symmetric group) and X is unknown. The relation of equations such as (2.10) to tensor equations is clear—a linear equation such as

$$LF_{\mu\nu\ldots\epsilon} = 0 \tag{2.11}$$

where L is a permutation operator on the n indices $\mu\nu\ldots\epsilon$, is equivalent to the substitutional equation

$$LX = 0 \tag{2.12}$$

where X is a permutation operator satisfying

$$F_{\mu\nu\ldots\bullet} = XF_{\mu\nu\ldots\epsilon}. \tag{2.13}$$

Thus, the solution of (2.11) is given by (2.13) where X satisfies (2.12). A complete account of the work of Young, and subsequent developments on the theory of equations such as (2.10), is given in Rutherford (1948). However, the general method for solving such equations, by the 'master indempotent', is very unwieldy except in the simplest cases. Nevertheless, for more complicated algebras of the type (1.2) this is presumably the only approach available at present. The problem may be simplified by applying such physical requirements as Lagrangian origin, and we hope to discuss such points in the future.

For a less trivial example of an algebra which is equivalent to a simple symmetry group, consider the sub-algebra of the causal algebra (1.4), defined by

$$\begin{aligned}
&(Y_{[4]}\, M)_{\mu\nu\rho\alpha} = 0 \\
&(Y^{(i)}_{[3,1]}\, M)_{\mu\nu\rho\alpha} = 0 \qquad i = 1, 2, 3 \\
&(Y^{(i)}_{[2,1,1]}\, M)_{\mu\nu\rho\alpha} = 0 \qquad i = 1, 2, 3 \\
&(Y^{(1)}_{[2,2]}\, M)_{\mu\nu\rho\alpha} = (Y_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}\, M)_{\mu\nu\rho\alpha} = 0.
\end{aligned} \tag{2.14}$$

In this case we have

$$\begin{aligned}
M_{\mu\nu\rho\alpha} &= \frac{1}{4!}\left(Y_{[4]} + 3\sum_{i=1}^{3} Y^{(i)}_{[3,1]} + 2\sum_{i=1}^{2} Y^{(i)}_{[2,2]} + 3\sum_{i=1}^{3} Y^{(i)}_{[2,1,1]} + Y_{[1,1,1,1]}\right) M_{\mu\nu\rho\alpha} \\
&= \frac{1}{4!}\left(2Y_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} + Y_{[1,1,1,1]}\right) M_{\mu\nu\rho\alpha}
\end{aligned} \tag{2.15}$$

by (2.14). It can be verified that the last expression has the symmetry

$$G_4 = \{e, (12)(34), (13)(24), (14)(23), (1)(2)(34)^{(-)}, (12)(3)(4)^{(-)}, (1324)^{(-)}, (1423)^{(-)}\} \tag{2.16}$$

(again, 1, 2, 3, 4 refer to the indices μ, ν, ρ, α). Conversely, suppose $M_{\mu\nu\rho\alpha}$ has the symmetry G_4, but does not satisfy any lower rank algebra. Thus

$$gM_{\mu\nu\rho\alpha}=\delta_g M_{\mu\nu\rho\alpha} \tag{2.17}$$

for $g\in G_4$, where δ_g is ± 1 as appropriate. (2.5) yields $n_4=21$ for the number of independent components of $M_{\mu\nu\rho\alpha}$ satisfying (2.17). This agrees with the number of independent components of $M_{\mu\nu\rho\alpha}$ satisfying the algebra (2.14), given by the solution (2.15) (20 for $Y_{\substack{13\\24}} M_{\mu\nu\rho\alpha}$, 1 for $Y_{\substack{\square\\\square\\\square\\\square}} M_{\mu\nu\rho\alpha}$). Also, every solution of (2.14) is a solution of (2.17). Thus, (2.14) and (2.17) are equivalent.

For the fifth rank products, applying G_4 to the two sets of four adjacent indices we find that the whole of S_5 is generated. In this case this is easy enough to verify directly, using a computer to do the calculations, if necessary, but in more complicated cases one could make use of knowledge of the subgroup structure of the permutation groups. Further, the fifth rank product changes sign under odd permutations (modulo third and first rank products), and so is antisymmetric in all pairs of indices. Thus, there are no independent fifth rank products (in four dimensions)—they all reduce, by the algebra (2.14).

The algebra (2.14) is therefore finite, and has 105 independent elements (4 for Γ_μ, 16 for $\Gamma_\mu\Gamma_\nu$, 64 for $\Gamma_\mu\Gamma_\nu\Gamma_\rho$, 20 for $Y_{\substack{13\\24}}\Gamma_\mu\Gamma_\nu\Gamma_\rho\Gamma_\alpha$ and 1 for $Y_{\substack{\square\\\square\\\square\\\square}}\Gamma_\mu\Gamma_\nu\Gamma_\rho\Gamma_\alpha$). Again, this algebra is not large enough to support a high-spin theory.

3. The algebra (1.4)

In this algebra the fourth rank product $\Gamma_{\mu\nu\rho\alpha}$ can be expressed in the form

$$\Gamma_{\mu\nu\rho\alpha}=\frac{1}{4!}\Big(2\sum_{i=1}^{2} Y^{(i)}_{\substack{\square\square\\\square\square}}+Y_{\substack{\square\\\square\\\square\\\square}}\Big)\Gamma_{\mu\nu\rho\alpha}+\Gamma\text{ products of 2nd rank} \tag{3.1}$$

(Cox 1976). The right-hand side has the symmetry

$$G_4=\{e,(12)(34),(13)(24),(14)(23)\} \tag{3.2}$$

and in the previous paper it was mistakenly assumed that this symmetry group was equivalent to the algebra (1.4). This is not so however, since $Y_{\square\square\square\square}$ is also invariant under G_4 and so:

$$M_{\mu\nu\rho\alpha}=\frac{1}{4!}\Big(Y_{\square\square\square\square}+2\sum_{i=1}^{2} Y^{(i)}_{\substack{\square\square\\\square\square}}+Y_{\substack{\square\\\square\\\square\\\square}}\Big)M_{\mu\nu\rho\alpha} \tag{3.3}$$

would be invariant under G_4, and the first equation of (1.4) need not be satisfied—i.e. the symmetry group G_4 acting on $M_{\mu\nu\rho\alpha}$ does not imply

$$(Y_{\square\square\square\square}\ M)_{\mu\nu\rho\alpha}=0. \tag{3.4}$$

It does however imply the remaining relations

$$(Y^{(i)}_{\substack{\square\square\square\\\square}}\ M)_{\mu\nu\rho\alpha}=0 \qquad (Y^{(i)}_{\substack{\square\square\\\square\\\square}}\ M)_{\mu\nu\rho\alpha}=0 \tag{3.5}$$

and so implies a more general algebra than (1.4). If we apply (2.5) to the group G_4 we find $n_4 = 76$, so that the algebra implied by G_4 has 76 independent fourth rank products. This agrees with the standard calculation for the number of independent components for the Young symmetrizers $(Y_{\square\square\square\square}M)_{\mu\nu\rho\alpha}$, $(Y^{(i)}_{\substack{\square\square\\\square\square}} M)_{\mu\nu\rho\alpha}$, $(Y_{\substack{\square\\\square\\\square\\\square}}M)_{\mu\nu\rho\alpha}$ in four dimensions, which yields $35+2\times 20+1=76$.

So the method given in the last section does not apply to the algebra (1.4). We now give an alternative argument, which in fact shows that the algebra is finite, but still not large enough for a good high-spin theory.

The algebra (1.4) is equivalent to the system

$$(pM)_{\mu\nu\rho\alpha} = M_{\mu\nu\rho\alpha} \qquad p \in G_4 \tag{3.6}$$

$$(Y_{\square\square\square\square} M)_{\mu\nu\rho\alpha} = 0 \tag{3.4}$$

since these are a consequence of (1.4) and conversely (3.6) implies (3.5) which together with (3.4) gives (1.4). Again, we can rewrite (3.6), (3.4) as

$$(p\Gamma)_{\mu\nu\rho\alpha} = \Gamma_{\mu\nu\rho\alpha} + \text{2nd rank products} \tag{3.7}$$

$$(Y_{\square\square\square\square}\,\Gamma)_{\mu\nu\rho\alpha} = \text{2nd rank products.} \tag{3.8}$$

In particular, putting all indices equal in (3.8) gives the minimal polynomial for each Γ_μ:

$$\Gamma_\mu^4 = \Gamma_\mu^2. \tag{3.9}$$

It is this part of the algebra which allows the reduction in the rank of the products—the remaining part merely allows rearrangement of the order of factors in the products. (3.9) implies that in any given product no more than three adjacent indices can be the same.

The equations (3.7) imply that for any given choice of indices $\mu\nu\rho\alpha$ we can always move any one of them, say μ, to the extreme left, and then a complete set of independent fourth rank products can be obtained from $\Gamma_{\mu\nu\rho\alpha}$, $\Gamma_{\mu\nu\alpha\rho}$, $\Gamma_{\mu\rho\nu\alpha}$, $\Gamma_{\mu\rho\alpha\nu}$, $\Gamma_{\mu\alpha\rho\nu}$, $\Gamma_{\mu\alpha\nu\rho}$, although not all of these need be independent (they will not be if, for example, two indices are the same). For example, the independent fourth rank products with all indices different can be taken as Γ_{1234}, Γ_{1243}, Γ_{1324}, Γ_{1342}, Γ_{1423}, Γ_{1432}. It can be verified that a fourth rank tensor, in four dimensions, with this property that any one index can be moved to the left, has 76 independent components, as it should have, by the equivalence of this symmetry to the algebra (3.5). The equation (3.4) implies another 35 independent relations, which further reduces the number of fourth rank products to 41. As well as the 'reduction equations' (3.9), (3.4) also contains further commutation rules allowing the indices to be further permuted (modulo lower rank products) in addition to G_4. However, it is not necessary to take account of these extra relations to show that the algebra (1.4) is not high-spin. It is sufficient, and much easier, to work with (3.6) and (3.9) only. Although this will overestimate the number of independent products in the algebra, the result will still be too small for a high-spin theory.

We first point out that the number of independent elements in a high-spin algebra for theories based on (1.1), with real Lagrangian origin, and manifestly covariant under the complete Lorentz group, is surprisingly large. As always, the manifest covariance of a first order equation such as (1.1) requires a number of auxiliary fields, and these, while playing no physical part in the theory, inflate the dimension of the Γ_μ matrices and therefore the size of their algebra. For example, the simplest spin-$\frac{3}{2}$ theory is one based on the representation $\mathscr{D}(\frac{1}{2}, 0)\oplus\mathscr{D}(0, \frac{1}{2})\oplus\mathscr{D}(\frac{1}{2}, 1)\oplus\mathscr{D}(1, \frac{1}{2})$. Up to equivalence in the Γ_μ there is only one good theory based on this representation, and the equation (1.1) (Gel'fand *et al* 1963, also see Hurley and Sudarshan 1975). This is the original spin-$\frac{3}{2}$ equation of Fierz–Pauli (Fierz and Pauli 1939, Gupta 1954), or its equivalent form, the Rarita–Schwinger (RS) equation (Rarita and Schwinger 1941). The matrices in this case are 16×16 and irreducible, so the algebra generated has dimension 256. However, this theory is well known to be acausal (Velo and Zwanzinger 1969). So any causal high-spin theory must be a little more complicated, and the simplest way this can be achieved is to double up the $\mathscr{D}(\frac{1}{2}, 0)$, $\mathscr{D}(0, \frac{1}{2})$ representations. This will require 20×20 Γ_μ matrices, and the Γ algebra must have 400 independent elements. However, even this theory is not entirely satisfactory. In a recent study of this representation, $2\mathscr{D}(\frac{1}{2}, 0)\oplus 2\mathscr{D}(0, \frac{1}{2})\oplus \mathscr{D}(\frac{1}{2}, 1)\oplus\mathscr{D}(1, \frac{1}{2})$, Hurley and Sudarshan (1975) have shown that a good theory (covariance under $\mathscr{L}_p$ and reflection, Lagrangian origin, unique mass and spin-$\frac{3}{2}$, *but not necessarily causal*), can only be achieved for either the RS theory (non-causal) or a theory in which the rank of the Γ algebra is five—i.e. the minimal equation is of fifth degree. Since our Γ algebra has rank four, the extended representation will still not take us out of the RS theory. However, Hurley and Sudarshan observe that theories with differing dynamics to the RS theory may be possible if the Γ algebra is reducible but indecomposable, using the extended representation. We do not know what this requires of our algebra (1.4), but we will be optimistic and suppose that a causal spin-$\frac{3}{2}$ theory may be possible with a Γ algebra of 400 elements. We will see that even this is too large for (1.4).

For good causal integer spin theories we will need even larger algebras—a brief survey of spin-2 theories based on (1.1) is given by Cox (1974) (see also Shamaly and Capri 1971, Frank 1973). So, any good causal high-spin theory based on (1.1) is obliged to have a Γ algebra with at least 400 independent elements.

First, consider an arbitrary rth rank product in the algebra (1.4), $r>4$:

$$\Gamma_{\mu_1\mu_2\ldots\mu_r}=\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_r}.$$

According to the symmetry (3.6) we can permute any index, in any adjacent four, to the extreme left position. By working through sets of four adjacent indices we can therefore pull any $r-3$ indices to the front in the product (modulo lower rank products), and furthermore these first $r-3$ indices can be permuted arbitrarily amongst themselves. Thus, the independent rth rank product can always be written in the form

$$\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_{r-3}}\Gamma_{\mu_{r-2}}\Gamma_{\mu_{r-1}}\Gamma_{\mu_r}$$

where $\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_{r-3}}$ are completely symmetric and $\Gamma_{\mu_{r-2}}\Gamma_{\mu_{r-1}}\Gamma_{\mu_r}$ are arbitrary for any choice of the indices, where the first $r-3$ can be chosen at will. It follows that no particular value of an index may occur more than three times, because then it would be possible to bring these equal indices adjacent and use (3.9) to reduce the rank of the product. The most we can do is have a product with all four different indices occurring three times, and such a product could always be written in the form $\Gamma_1^3\Gamma_2^3\Gamma_3^3\Gamma_4^3$. Thus the maximum rank of independent product, subject to (3.6) and (3.9), is twelve, and there is only one of these. We can now work down, counting the independent lower rank products, subject to (3.6) and (3.9). In the following, when the indices $\mu\nu\rho\alpha$ occur they are to be regarded as distinct. Also, we denote Γ_μ^k by μ^k, etc.

11th rank, 4 products
$\mu^2\nu^3\rho^3\alpha$

10th rank, 10 products
$\mu\nu^3\rho^3\alpha^3(4)$, $\mu^2\nu^2\rho^3\alpha^3(6)$

9th rank, 20 products
$\nu^3\rho^3\alpha^3(4)$, $\mu\nu^2\rho^3\alpha^3(12)$, $\mu^2\nu^2\rho^2\alpha^3(4)$

8th rank, 33 products
$\nu^2\rho^3\alpha^3(12)$, $\mu\nu\rho^3\alpha^3(6)$, $\mu\nu^2\rho^2\alpha^3(12)$, $1^22^23^24^2, 1^22^23434, 1^22^234^23(3)$

7th rank, 48 *products*
$\nu\rho^3\alpha^3(12)$, $\nu^2\rho^2\alpha^3(12)$, $\mu\nu\rho^2\alpha^3(12)$, $\mu\nu^2\rho^2\alpha^2, \mu\nu^2\rho\alpha\rho\alpha, \mu\nu^2\rho\alpha^2\rho(12)$

6th rank, 64 products
$\rho^3\alpha^3(6)$, $\nu\rho^2\alpha^3(24)$, $\mu\nu\rho\alpha^3(4)$, $\nu^2\rho^2\alpha^2, \nu^2\rho\alpha\rho\alpha, \nu^2\rho\alpha^2\rho(12)$, $\mu\nu\rho^2\alpha^2, \mu\nu\rho\alpha\rho\alpha,$
$\mu\nu\rho\alpha^2\rho(18)$

5th rank, 84 products
$\rho^2\alpha^3(12)$, $\nu\rho\alpha^3(12)$, $\nu\rho^2\alpha^2, \nu\rho\alpha\rho\alpha, \nu\rho\alpha^2\rho(36)$, $\mu^2\nu\rho\alpha(24)$.

Fourth and lower rank products for (1.4) are easily found directly—41 fourth rank, 64 third rank, 16 second rank and 4 first rank. As we have said, the above results overestimate the size of the algebra (1.4), using as they do only the relation (3.6) and (3.9), but even so they only yield a total maximum of 389 for the algebra (1.4), which is too small for a high-spin theory. The algebra (1.4) cannot therefore support a causal high-spin theory. This is even more true of any sub-algebra of (1.4)—in particular, the algebra considered in § 2.

4. Conclusion

A sufficient condition for causality in the external field problem for minimal electromagnetic coupling with a field described by (1.1) is that the principle part of the reduced equation of motion be Klein–Gordon. The only well known theories which satisfy this requirement are the Dirac spin-$\frac{1}{2}$ and DK spin, 0,1 theories, and it is natural to look for high-spin theories. In an earlier paper we have given a simple algebraic technique for obtaining theories satisfying this sufficient condition, in the form of Young symmetrizer algebra, and there gave the example of a fourth rank causal algebra, incorrectly stating it to be infinite. In this paper we give a more thorough analysis of this algebra, and find that it is in fact finite, but not large enough to accommodate a high-spin theory.

We also describe a method of counting the independent elements of a special class of Young symmetrizer algebras. The method applies to algebras which are equivalent to a simple symmetry group operating on a given rank of product. It involves finding the consequent subgroup of S_r under which the rth rank product in the algebra is either invariant or changes sign (modulo lower rank products). This method is not essential in this paper—it does not apply to the algebra (1.4)—but it may be helpful in the study of higher rank algebras, which we hope to describe in a later publication.

It may be that there are no 'nicely interacting' causal high-spin theories—for example, if it turns out that the algebras generated by the Klein–Gordon principal part requirement are too small to accommodate spin greater than one. The only alternative then would be that the interaction part of the reduced equation of motion, in which the constraints have been eliminated, contributes second derivative terms in such a way that the propagation remains causal, and perhaps even this may be impossible for high-spin.

Note that in the approach adopted here, it was not necessary to have detailed knowledge of the algebras concerned to eliminate the possibility of high-spin—we merely needed an upper bound on the size of the algebra which was less than that required for the simplest high-spin theory. This idea may extend to higher rank algebras. On the other hand, if we obtain an algebra which is apparently large enough for high-spin, we would then have to analyse the algebra thoroughly to see if it did indeed have non-trivial representations giving good high-spin theories, and this would be a very difficult task. In this paper, we have not pursued the causal algebra or its sub-algebra further because they can only yield spin-1 at the most and our interest is in whether high-spin theories exist.

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